RATIONAL CONNECTEDNESS OF LOG Q-FANO VARIETIES

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Abstract. In this paper, we give an affirmative answer to a conjecture in the Minimal Model Program. We prove that log Q-Fano varieties are rationally connected. We also study the behavior of the canonical bundles under projective morphisms

§1. Log Q-Fano varieties are rationally connected

Let X be a log Q-Fano variety, i.e, if there exists an effective Q-divisor D such that the pair (X, D) is Kawamata log terminal (*klt*) and $-(K_X + D)$ is nef and big. By a result of Miyaoka-Mori [15], X is uniruled. A standard conjecture ([10], [12], [13], [16]) predicts that X is actually rationally connected. In this paper we apply the theory of weak (semi) positivity of the direct images of (log) relative dualizing sheaves $f_*(K_{X/Y} + \Delta)$ (which has been developed by Fujita, Kawamata, Kollár, Viehweg and others) to show that a log Q-Fano variety is indeed rationally connected.

Remark: The rational connectedness of smooth Fano varieties was established by Campana [1] and Kollár-Miyaoka-Mori [12]. However their approach relies heavily on the (relative) deformation theory which seems quite difficult to extend to the singular case.

Theorem 1. *Let X be a log Q-Fano variety. Then X is rationally connected, i.e., for any two closed points $x, y \in X$ there exists a rational curve C which contains x and y.*

Remark: When $\dim(X) \leq 3$ and $D = 0$, this was proved by Kollár-Miyaoka- Mori in [13]. On the other hand, the result is false for log canonical singularities (see 2.2 in [13]).

As a corollary, we can show the following result which was obtained by S. Takayama [16].

Corollary 1. *Let X be a log Q-Fano variety.Then X is simply connected.*

Proof. $\pi_1(X)$ is finite by Theorem 1 and a result of F. Campana [2]. On the other hand, $h^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$ by Kawamata-Viehweg vanishing ([4], [9]). Thus we have $\chi(X, \mathcal{O}_X) = 1$ and hence X must be simply connected.

Our Theorem 1 is a consequence of the following proposition.

Proposition 1. *Let X be a log Q-Fano variety and let $f : X - -- > Y$ be a dominant rational map, where Y is a projective variety. Then Y is uniruled if $dim Y > 0$.*

Let us assume Proposition 1. Let X' be a resolution of X, then X' is uniruled. There exists a nontrivial maximal rationally connected fibration $f : X' - -- > Y$ ([1], [12]). By a result of Graber-Harris-Starr [5], Y is not uniruled. However Proposition 1 tells us that Y must be a point and hence X is rationally connected.

The general strategy for proving Proposition 1 is as follows. By a result of Miyaoka-Mori [15], it suffices to construct a covering family of curves C_t on Y with $C_t \cdot K_Y < 0$ for every t. To this end, we apply the positivity theorem of the direct images of (log) relative dualizing sheaves to $f : X - - > Y$. We show that there exist an ample Q-divisor H on Y and an effective Q-divisor D on X such that $-K_Y = D + f^*H$ (modulo some exceptional divisors). Now let $C_t = f(F_t)$, where F_t are the general complete intersection curves on X. We have a covering family of curves C_t on Y with $C_t \cdot K_Y < 0$ for every t.

Before we start to prove Proposition 1, let us first give some related definitions. Also the proof of Proposition 1 depends heavily on Kawamata's paper [8] and Viehweg's paper [17].

We work over the complex number field $\mathbb{C}$ in this paper.

Definition 1 [9]. *Let X be a normal projective variety of dimension n and K_X the canonical divisor on X. Let $D = \sum a_i D_i$ be an effective Q-divisor on X, where D_i are distinct irreducible divisors and $a_i \geq 0$. The pair (X, D) is said to be Kawamata log terminal (klt) (resp. log canonical) if $K_X + D$ is a Q-Cartier divisor and if there exists a desingularization (log resolution) $f : Z \to X$ such that the union F of the exceptional locus of f and the inverse image of the support of D is a divisor with normal crossing and*

$$K_Z = f^*(K_X + D) + \sum_i e_j F_j,$$

with $e_j > -1$ (resp. $e_j \geq -1$). X is said to be Kawamata log terminal (resp. log canonical) if so is $(X, 0)$.

Definition 2. *Let X be a normal projective variety of dimension n and K_X the canonical divisor on X. We say X is a Q-Gorenstein variety if there exists some integer $m > 0$ such that mK_X is a Cartier divisor. A Q-Cartier divisor D is said to be nef if the intersection number $D \cdot C \geq 0$ for any curve C on X. D is said to be big if the Kodaira-Iitaka dimension $\kappa(D)$ attains the maximum $dimX$.*

The following lemma due to Raynaud [19] is quite useful:

Lemma 1. *Let $g : T \to W$ be surjective morphism of smooth varieties. Then there exists a birational morphism of smooth variety $\tau : W' \to W$ and a desingularization $T' \to T \times_W W'$, such that the induced morphism $g' : T' \to W'$ has the following property: Let B' be any divisor of T' such that $codim(g'(B')) \geq 2$. Then B' lies in the exceptional locus of $\tau : T' \to T$.*

Proof of Proposition 1. By the Stein factorization and desingularizations, we may assume that Y is smooth. Resolving the indeterminacy of f and taking a log resolution, we have a smooth projective variety Z and the surjective morphisms $g : Z \to Y$ and $\pi : Z \to X$,

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & X \\ \Big\downarrow{\scriptstyle g} & & \\ Y & & \end{array}$$

such that

$$K_Z = \pi^*(K_X + D) + \sum_i e_i E_i$$

with $e_i > -1$, where $\sum E_i$ is a divisor with normal crossing.

Since $-(K_X + D)$ is nef and big, by Kawamata base-point free theorem, there exists an effective Q-divisor A on X such that $-(K_X + D) - A$ is an ample Q-divisor. Thus we can choose another ample Q-divisor H on Y and an effective Q-divisor $\Delta = \sum_i \delta_i E_i$ on Z (with small $\delta \geq 0$ and E_i are π-exceptional if $\delta_i > 0$) such that $-\pi^*(K_X + D + A) - \Delta - g^*(H) = L$ is again an ample Q-divisor on Z. We may also assume that $\mathrm{Supp}(\Delta + \cup_i E_i + A)$ is a divisor with simple normal crossing and the pair $(X, L + \Delta + \sum_i \{-e_i\} E_i + A)$ is klt. Thus we have

$$K_{Z/Y} + \sum_i \epsilon_i E_i \sim_Q \sum_i m_i E_i - g^*(K_Y + H)$$

where $m_i = \lceil e_i \rceil$ are non-negative integers ($\{,\}$ is the fractional part and $\lceil, \rceil$ is the round up). Also E_i on the left side contain components of L, A and Δ with $0 \leq \epsilon_i < 1$.

We follow closely from Kawamata's paper [8].

After further blowing-ups if necessary, and by Lemma 1, we may assume that:

(1) There exists a normal crossing divisor $Q = \sum_l Q_l$ on Y such that $g^{-1}(Q) \subset \sum_i E_i$ and g is smooth over $Y \setminus Q$.
(2) If a divisor W on Z with $\operatorname{codim}(g(W)) \geq 2$, then W is π-exceptional.

Let $D = \sum_i (\epsilon_i - m_i) E_i = \sum_i d_i E_i = D^h + D^v$, where

(1) $g : D^h \to Y$ is surjective and smooth over $Y \setminus Q$ (we say D^h is g-horizontal)
(2) $g(D^v) \subset Q$ (we say D^v is g-vertical [8]).

Notice that here besides those E_i from the log resolution π, D also contains the components of $\operatorname{Supp}(L + \Delta + A)$. Nevertheless $d_i < 1$ for all i.

We have two cases:

(1): Every π-exceptional divisor E_i with $d_i < 0$ is g-vertical. In this case, the natural homomorphism $\mathcal{O}_Y \to g_*(\lceil -D \rceil)$ is surjective at the generic point of Y.

Let
$$g^* Q_l = \sum_j w_{lj} E_j$$
and
$$a_j = \frac{d_j + w_{lj} - 1}{w_{lj}} \text{ if } g(E_j) = Q_l$$
and
$$b_l = \{\max\{a_j\} : g(E_j) = Q_l\}.$$

Let $N = \sum_l b_l Q_l$ and $M = -H - K_Y - N$. Then by a result of Kawamata [8, Theorem 2], M is nef.

On the other hand, $g^* N = F + G$, where

(1) $\operatorname{Supp}(F)$ is π-exceptional (from those Q_l with $g^* Q_l = \sum_j w_{lj} E_j$ and each E_j is π-exceptional).
(2) G is effective (from those Q_l with $g^* Q_l = \sum_j w_{lj} E_j$ and at least one E_j has coefficient $d_j \geq 0$).

Now let C be a general complete intersection curve on Z such that C does not intersect with $\operatorname{Supp}(F)$ (e.g., pull-back a general complete intersection curve from X). Then
$$g_*(C) \cdot (-K_Y - H) = C \cdot g^*(-K_Y - H) = C \cdot g^*(M + N) \geq 0$$
and hence $g_*(C) \cdot K_Y \leq g_*(C) \cdot (-H) < 0$. Since the family of the curves $g(C)$ covers a Zariski open set of Y, by [15] Y is uniruled.

(2): Some π-exceptional divisor is g-horizontal, in particular, D^h is not zero.

We have $K_{Z/Y} + D \sim_Q -g^*(K_Y + H)$. By the stable reduction theorem and the covering trick ([8],[17]), there exists a finite morphism $p : Y' \to Y$ such that $Q' = \mathrm{Supp}(p^*Q)$ is a normal crossing divisor and the induced morphism $g' : Z' \to Y'$ from a desingularization $Z' \to Z \times_Y Y'$ is semistable over $Y' \setminus B$ with $\mathrm{codim}(B) \geq 2$. Let $Z' \to Z$ be the induced morphism.

$$\begin{array}{ccccc} Z' & \xrightarrow{q} & Z & \xrightarrow{\pi} & X \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \\ Y' & \xrightarrow{p} & Y & & \end{array}$$

We can write

$$K_{Z'/Y'} + D' \sim_Q g'*p^*(-K_Y - H), \text{where } D' = \sum_{j'} d'_{j'} E'_{j'}.$$

The coefficients $d'_{j'}$ can be calculated as follows [8]:

(1) If $E'_{j'}$ is g'-horizontal and $q(E'_{j'}) = E_j$, then $d'_{j'} = d_j$.
(2) If $E'_{j'}$ is g'-vertical with $q(E'_{j'}) = E_j$ and $g'(E'_{j'}) = Q'_{l'}$, then $d'_{j'} = e_{j'}(d_j + w_{lj} - 1)$, where $e_{l'}$ is the ramification index of q at the generic point of $E'_j \to E_j$.
(3) We are not concerned with those E'_j such that $g'(E'_j) \subset B$.

Note: In Kawamata's paper [8], he replaced D by $D - g^*N$. Thus all the coefficients there are ≤ 0. However, here we do not make such replacement and in our case some coefficients $d_i = \epsilon_i > 0$

Therefore, by the standard trick (keep the fractional part on the left side and the integral part on the right side, also blow-up Z' if necessary). We have

$$K_{Z'/Y'} + \sum_{j'} \epsilon'{}_{j'} E'_{j'} \sim_Q \sum_{k'} n_{k'} E'_{k'} - {g'}^* p^*(K_Y + H) - V + G$$

such that

(1) $\sum_{k'} n_{k'} E'_{k'}$ is Cartier, $\mathrm{Supp}(\sum_{k'} E'_{k'})$ is $q \circ \pi$-exceptional and $g'(\sum_{k'} n_{k'} E'_{k'})$ is not contained in B.
(2) V is an effective Cartier divisor which is g'-vertical (from those g-vertical E_i with $d_i = \epsilon_i \geq 0$ and $d'_{j'} = e_{j'}(d_j + w_{lj} - 1) \geq 1$ for some l.)

(3) G is also Cartier and $q \circ \pi$-exceptional (from those E'_j with $g'(E'_j) \subset B$).
(4) $\sum_{j'} d_{j'} E'_{j'}$ remains on the left side, where $E'_{j'}$ are g'-horizontal with $0 < d_{j'} = d_j = \epsilon_j < 1$
(5) If $E'_{k'}$ is g'-horizontal, then $n_{k'} \geq 0$.
(6) $(Z', \sum_{j'} \epsilon'_{j'} E'_{j'})$ is klt.

There also exists a cyclic cover [9] $p' : Y'' \to Y'$ such that Y'' is smooth and $p'^* p^*(H) = 2H'$, where H' is an ample Cartier divisor. Since H is an ample Q-divisor, we can choose the covering p' in such way that the ramification locus $R_{p'}$ of p' intersects Supp$Q$ and B transversely. Let $g'' : Z'' \to Y''$ be the induced morphism from a desingularization $Z'' \to Z' \times_{Y'} Y''$.

$$\begin{array}{ccccccc} Z'' & \xrightarrow{q'} & Z' & \xrightarrow{q} & Z & \xrightarrow{\pi} & X \\ {\scriptstyle g''}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \\ Y'' & \xrightarrow{p'} & Y' & \xrightarrow{p} & Y & & \end{array}$$

Since g' is semistale over $Y' \setminus B$, we have $q'{*}K_{Z'/Y'} = K_{Z''/Y''}$ over $Y'' \setminus p'^{-1}(B)$. Thus again we can write

$$K_{Z''/Y''} + \sum_{j''} \epsilon''_{j''} E''_{j''} \sim_Q \sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - 2g''^* H' - V' + G'$$

where

(1) $\sum_{k''} n_{k''} E''_{k''}$ is Cartier and Supp$(\sum_{k''} E''_{k''})$ is $q' \circ q \circ \pi$-exceptional.
(2) V' is an effective Cartier divisor which is g''-vertical.
(3) G' is also Cartier and $q' \circ q \circ \pi$-exceptional (since codim$g''(G') \geq 2$).
(4) If $E''_{k''}$ is g''-horizontal, then $n_{k''} \geq 0$.
(5) $(Z'', \sum_{j''} \epsilon''_{j''} E'_{j''})$ is klt and $\sum_{j''} \epsilon''_{j''} E''_{j''}$ is Q-linearly equivalent to a Cartier divisor.

Since all the g''-horizontal divisors $E''_{k''}$ have non-negative coefficients,

$$g''_*(\sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - 2g''^* H' - V' + G')$$

is not a zero sheaf. Let $\omega = \sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - V' + G'$.

Applying the results of Kollár [11], Viehweg [17, Lemma 5.1] and Kawamata [7, Theorem 1.2], we may assume that

$$g''_*(\omega - 2g''^* H') = g''_*(\omega) \otimes \mathcal{O}_{Y''}(-2H')$$

is torsion free and weakly positive over Y''.

Note: In [7], Kawamata proved that in fact (after blow-up Y'' further) we may assume that ${g''}_*(\omega)$ is locally free and semipositive. However the weak positivity is sufficient in our case.

By the weak positivity of ${g''}_*(\omega) \otimes \mathcal{O}_{Y''}(-2H')$, we have

$$\hat{\mathcal{S}}^n {g''}_*(\omega) \otimes \mathcal{O}_{Y''}(-2nH' + nH')$$

is generically generated by its global sections over Y'' for some $n > 0$, where $\hat{\mathcal{S}}^n$ denotes the reflexive hull of $\mathcal{S}^n$.

We have a natural homomorphism

$$S^n g''^* {g''}_*(\omega) \otimes g''^* \mathcal{O}_{Y''}(-nH') \to \omega^n \otimes g''^* \mathcal{O}_{Y''}(-nH')$$

By the torsion freeness of ${g''}_*(\omega)$, there exists an open set $U \subset Y''$ with $\operatorname{codim}(Y'' \setminus U) \geq 2$ such that

$$\hat{\mathcal{S}}^n {g''}_*(\omega) \otimes \mathcal{O}_{Y''}(-nH')|_U = \mathcal{S}^n {g''}_*(\omega) \otimes \mathcal{O}_{Y''}(-nH')|_U$$

and hence

$$g''^* \hat{\mathcal{S}}^n {g''}_*(\omega) \otimes g''^* \mathcal{O}_{Y''}(-nH')|_W = S^n g''^* {g''}_*(\omega) \otimes g''^* \mathcal{O}_{Y''}(-nH')|_W$$

where $W = g''^{-1}(U)$. If $B' = Z'' \setminus W$, then B' is g''-exceptional. Since $g''^* \hat{\mathcal{S}}^n {g''}_*(\omega) \otimes g''^* \mathcal{O}_{Y''}(-nH')$ is also generically generated by its global sections over Z'', there is a non trivial morphism

$$\bigoplus \mathcal{O}_{Z''}|_W \to \omega^n \otimes g''^* \mathcal{O}_{Y''}(-nH')|_W$$

i.e., $\omega^n \otimes g''^* \mathcal{O}_{Y''}(-nH')$ admits a meromorphic section which has poles only along B'. Thus we may choose some large integer k such that $\omega^n \otimes g''^* \mathcal{O}_{Y''}(-nH') + kB'$ has a holomorphic section, i.e.,

$$n(\sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - V' + G') - n g''^* H' + kB'$$

is effective.

Again as before, we can choose a family of general complete intersection curves C on Z'' such that C does not intersect with the exceptional locus of $Z'' \to X$ (such as ${E''}_{k''}$, B' and G'). Thus ${g''}_*(C) \cdot p'^* p^*(K_Y) \leq {g''}_*(C) \cdot (-H') < 0$ and hence Y is uniruled [15]. q.e.d.

§2.The behavior of the canonical bundles under projective morphisms

Let X and Y be two projective varieties and $f : X \to Y$ be a surjective morphism. Assume that the Kodaira dimension $\kappa(X) \leq 0$. In general, it is almost impossible to predict the Kodaira dimension of Y. The following example shows that even when dimY = 1, we have no control of the genus of Y:

Example. *([6], [13]): Let C be a smooth curve of arbitrary genus g and A be an ample line bundle on C such that $degA > 2degK_C$. Let $S = Proj_C(\mathcal{O}(A) \oplus \mathcal{O}_C)$ be the projective space bundle associated to the vector bundle $\mathcal{O}(A) \oplus \mathcal{O}_C$. Then $K_S = \pi^*(K_C + A) - 2L$, where L is the tautological bundle and $\pi : S \to C$ is the projection. An easy computation show that $-K_S$ is big (i.e, $h^0(S, -mK_S) \approx c \cdot m^2$ for $m \gg 0$ and some $c > 0$). In particular, the Kodaira dimension $\kappa(S) = -\infty$. However, the genus of C could be large.*

On the other hand, it is not hard to find the following facts:

(1) Let $D = -K_S = 2(L - \pi^*A) + \pi^*(A - K_C)$, then D is effective and the pair (S, D) is not log canonical [9].
(2) $-K_S$ is not nef, i.e, there exists some curve B (e.g., choose B to be the section corresponding to $\mathcal{O}(A) \oplus \mathcal{O}_C \to \mathcal{O}_C \to 0$) on S such that $(-K_S) \cdot B < 0$.
(3) For any integer $m > 0$, the linear system $-mK_S$ contains some fixed component (e.g., $m(L - \pi^*A)$) which dominates C.

In view of the above example, we give a few sufficient conditions which guarantees nice behavior of the Kodaira dimension (and the canonical bundle).

Theorem 2. *Let $f : X \to Y$ be a surjective morphism. Assume that $D \equiv -K_X$ is an effective Q-divisor and the pair (X, D) is log canonical. Moreover assume that Y is normal and Q-Gorenstein. Then either Y is uniruled or K_Y is numerically trivial. (In particular, $\kappa(Y) \leq 0$.)*

Corollary 2. *Let $f : X \to Y$ be a surjective morphism. Assume that X is log canonical and K_X is numerically trivial (e.g., a Calabi-Yau manifold). Moreover assume that Y is normal and Q-Gorenstein. Then either Y is uniruled or K_Y is numerically trivial.*

Theorem 3. *Let $f : X \to Y$ be a surjective morphism. Assume that $-K_X$ is nef and X is smooth. Moreover assume that Y is normal and Q-Gorenstein. Then either Y is uniruled or K_Y is numerically trivial.*

Remark: Theorem 3 was proved in [18] (in particular, the result solved a conjecture proposed by Demailly, Peternell and Schneider [3]). However, the proof given there was incomplete (as pointed out to me by Y. Kawamata. I wish to thank him). The problem lies in Proposition 1 in [18], the point is that the nefness in general is not preserved under the deformations (mod p reductions in our case). We shall present a new proof of this proposition (see Proposition 2).

Theorem 4. *Let $f : X \to Y$ be a surjective morphism. Assume that there exists some integer $m > 0$ such that $-mK_X$ is effective and has no fixed locus which dominates Y. Moreover assume that X is log canonical and Y is normal and Q-Gorenstein. Then either Y is uniruled or K_Y is numerically trivial.*

As an immediate consequence, we can show the following result about the Albanese maps.

Corollary 3. *Let X be a smooth projective variety. Then the Albanese map $Alb_X : X \to Alb(X)$ of X is surjective and has connected fibers if X satisfies one of the following conditions:*

(1) *$D \equiv -K_X$ is an effective Q-divisor and the pair (X, D) is log canonical ("$\equiv$" means numerically equivalent).*
(2) *$-K_X$ is nef .*
(3) *There exists some integer $m > 0$ such that $-mK_X$ is effective and has no fixed component which dominates $Alb(X)$.*

The main ingredients of the proofs are the Minimal Model Program (in particular, a vanishing theorem of Esnault-Viehweg, Kawamata and Kollár plays an essential role), and the deformation theory. It is interesting to notice that the proofs of Theorem 2 and Theorem 3 are completely different in nature.

The following vanishing theorem of Esnault-Viehweg, Kawamata and Kollár is important to us:

Vanishing Theorem [4], [22]. *Let $f : X \to Y$ be a surjective morphism from a smooth projective variety X to a normal variety Y. Let L be a line bundle on X such that $L \equiv f^*M + D$, where M is a Q-divisor on Y and (X, D) is Kawamata log terminal. Let C be a reduced divisor without common component with D and $D + C$ is a normal crossing divisor. Then*

(1) *$f_*(K_X + L + C)$ is torsion free [20].*
(2) *Assume in addition that M is nef and big. Then $H^i(Y, \mathcal{R}^j f_*(K_X + L + C)) = 0$ for $i > 0$ and $j \geq 0$.*

Remark: The $C = 0$ case was done by Esnault-Viehweg, Kawamata and Kollár [11]. The generalized version given here essentially was proved by Esnault-Viehweg in [4] and by Fujino in [22]. I thank Professor Viehweg for informing on the matter. C. Hacon pointed out an inaccuracy and informed me of the reference [22]. I would like to thank him. Below, we give an outline of the proof which was provided to me by E. Viehweg.

Sketch of the proof. By [4, 5.1 and 5.12], we have an injective morphism

$$0 \to H^j(X, K_X + L + C) \to H^j(X, K_X + L + C + B)$$

for any j, where $B = f^*(F)$ for some divisor F on Y. If we choose F to be a very ample divisor, we have the exact sequence:

$$0 \to \mathcal{R}^j f_*(K_X+L+C) \to \mathcal{R}^j f_*(K_X+L+C+B) \to \mathcal{R}^j f_*(K_B+L+C) \to 0.$$

By induction on $\dim Y$ and the Leray-spectral sequence associated with $\mathcal{R}^j f$, we can prove the result (see [4] for details). q.e.d.

Proof of Theorem 2. : Let $g : Z \to X$ be a log resolution and let $\pi = f \circ g$. Then

$$K_Z = g^*(K_X + D) + \sum a_i E_i, \text{ where each } a_i \geq -1.$$

We can rewrite $\sum a_i E_i = \sum b_j E_j + \sum c_k E_k + \sum d_l E_l$ where $b_j \geq 0$, $0 > c_k > -1$ and $d_l = -1$.

Let C be a general complete intersection curve on Y and $W = \pi^{-1}(C)$. We have

$$K_Z = K_{Z/Y} + \pi^* K_Y \text{ and } K_{Z/Y}|_W = K_{W/C}$$

Thus

$$K_W + \pi^*(K_Y|_C) + \sum -c_k E_k|_W + \sum \{-b_j\} E_j|_W + \sum E_l|_W \equiv \pi^* K_C + \sum \lceil b_j \rceil E_j|_W$$

Let us assume that $K_Y \cdot C > 0$. Since $(K_W, \sum -c_k E_k|_W + \sum \{-b_j\} E_j)$ is Kawamata log terminal and $\sum \lceil b_j \rceil E_j|_W$ is exceptional, the Vanishing Theorem yields

$$H^1(C, \pi_*(\pi^* K_C + \sum \lceil b_j \rceil E_j|_W)) = H^0(C, \mathcal{O}_C) = 0,$$

a contradiction. So we must have $K_Y \cdot C \leq 0$. If $K_Y \cdot C < 0$, Y is uniruled by [15]. If $K_Y \cdot C = 0$, K_Y is numerically trivial by Hodge index theorem. q.e.d.

Proof of Theorem 3. Let us first establish the following proposition (Proposition 1 in [18]).

Proposition 2. *Let $\pi : X \to Y$ be a surjective morphism between smooth projective varieties over $\mathbb{C}$. Then for any ample divisor A on Z, $-K_{X/Y} - \delta\pi^*A$ is not nef for any $\delta > 0$.*

Proof of Proposition 2. We shall give a new proof of this proposition by modifying the arguments we used before [18]. Again, the main idea and method comes from [14].

Let $C \subset X$ be a general smooth curve of genus $g(C)$ such that $C \nsubseteq \mathrm{Sing}(\pi)$. Let $p \in C$ be a general point and $B = \{p\}$ be the base scheme. Denoting by $\nu : C \to X$ the embedding of C to X. Let $D_Y(\nu, B)$ be the Hilbert scheme representing the functor of the relative deformation over Y of ν. Then by [14], we have

$$\mathrm{dim}_\nu D_Y(\nu, B) \geq -\nu_*(C) \cdot K_{X/Y} - g(C) \cdot \mathrm{dim}X$$

Suppose now that $-K_{X/Y} - \delta\pi^*A$ is nef for some $\delta > 0$. Let H be an ample divisor on X and $\epsilon > 0$ be a small number, we may assume that

$$\nu_*(C) \cdot (\delta\pi^*A - \epsilon H) > 0.$$

Then

$$\mathrm{dim}_\nu D_Y(\nu, B) \geq \nu_*(C) \cdot (\delta\pi^*A - \epsilon H) - g(C) \cdot \mathrm{dim}X$$

Since $-K_{X/Y} - \delta\pi^*A + \epsilon H$ is an ample divisor on X and the ampleness is indeed an open property in nature. By the method of modulo p reductions [14], after composing ν with suitable Frobenius morphism if necessary, we can assume that there exists another morphism [14] $\nu' : C \to X$ such that

(1) $\mathrm{deg}\nu'_*(C) < \mathrm{deg}\nu_*(C)$, where $\mathrm{deg}\nu_*(C) = \nu_*(C) \cdot H$.
(2) $\pi \circ \nu' = \pi \circ \nu$.

However, we have

$$\nu'_*(C) \cdot (\delta\pi^*A - \epsilon H) > \nu_*(C) \cdot (\delta\pi^*A - \epsilon H)$$

by (1) and (2). This guarantees the existence of a non-trivial relative deformation of ν'. Since $\mathrm{deg}\nu'_*(C) < \mathrm{deg}\nu_*(C)$, this process must terminate, which is absurd. q.e.d.

Proof of Theorem 3 continued: We keep the same notations as in Theorem 2. Let $-K_{W/C} = -K_X|_W + f^*(K_Y|_C)$, where C is a general complete intersection curve on Y and $W = f^{-1}(C)$. Applying Proposition 2, we deduce that $K_Y \cdot C \leq 0$ and we are done. q.e.d.

Proof of Theorem 4. Replacing X by a suitable resolution if necessary, we may assume that $-pK_X = L + N$ for some positive integer p, where $|L|$ is

base-point free and N is the fixed part. Multiplying both sides by some large integer m, we can write $-K_X = \epsilon L_m + N_m$ as Q-divisors, where $\epsilon > 0$ is a small rational number. We may again assume that the linear system L_m is base-point free and N_m is the fixed locus. The point is that $(X, \epsilon L_m)$ is log canonical. If the fixed locus does not dominate Y, we can choose a general complete intersection curve C on Y such that C only intersects $f(\mathrm{Supp}(N_m))$ at some isolated points. Using the same notations as before, we have

$$K_W + f^*(K_Y|_C) + \epsilon L_m|_W \equiv f^*K_C + E - N_m|_W$$

where E is exceptional. If $C \cdot K_Y > 0$, then by the Vanishing Theorem

$$H^1(C, f_*(K_W + f^*(K_Y|_C) + \epsilon L_m|_W + \{N_m\})) = H^0(C, -f_*(-\lfloor N_m|_W \rfloor)) = 0$$

Since $\mathrm{Supp}(N_m)$ is contained in some fibers of f, we reach a contradiction. The remaining arguments are exactly the same as in the proof of Theorem 2. q.e.d.

Proof of Corollary 3. Let $X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Alb}(X)$ be the Stein factorization of Alb_X. Then by Theorem 2-4, we conclude that $\kappa(Y) = 0$. We may assume that Y is smooth, otherwise we can take a desingularization Y' of Y. This will not affect our choices for the general curve C (since Y is smooth in codimension 1). Therefore $\mathrm{Alb}(X)$ must be an abelian variety and hence Alb_X is surjective and has connected fibers (see [18] for details). q.e.d.

Remark: The notion of special varieties was introduced and studied by F. Campana in [20]. He also conjectured that compact Kähler manifolds with $-K_X$ nef are special. S. Lu [21] proved the conjecture for projective varieties. In particular, if X is a projective variety with $-K_X$ nef and if there is a surjective map $X \to Y$. Then $\kappa(Y) \leq 0$. Our focus however, is on the uniruledness of Y.

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